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Average Value of $|K_2(\mathcal{O})|$ in Function Fields*

MICHAEL ROSEN

Mathematics Department, Brown University, Providence, Rhode Island 02912
E-mail: michael_rosen@brown.edu

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Let F be a finite field with q elements, $A = F[T]$ the polynomial ring over F, and $K = F(T)$. If $m \in A$ is a square-free polynomial, we denote by $\mathcal{O}_m$ the integral closure of A in $k(\sqrt{m})$. In this paper we determine, roughly speaking, the average value of the size of the groups $K_2(\mathcal{O}_m)$ as m varies of all square-free polynomials of a fixed degree M. The answer is a certain constant times $q^{3(M/2)}$ plus an error term of order q^M. The constant is determined precisely. © 1995 Academic Press, Inc.

1

In [1], the author and Jeff Hoffstein investigated the average value of L-functions $L(s, \chi)$ in quadratic function fields. By evaluating at $s = 1$ and using the analogue of Dirichlet's theorem which was first proven in function fields by Artin, we were able to give average value results for class numbers. As a very special case of conjectures of Lichtenbaum [2], since generalized by Beilinson, the special values $L(-n, \chi)$ at negative integers are related to the torsion in the algebraic K groups $K_m(\mathcal{O})$, $m \geq 0$, where $\mathcal{O}$ is the "ring of integers" in the field under consideration. In certain cases these conjectures have been verified. By working at $s = 2$ and using results of Tate and Quillen, we will give average value results for the size of the groups $K_2(\mathcal{O})$. For the exact result, see the statement of the Main Theorem in Section 3.

We begin by setting notation. Let F be a finite field with q elements, $A = F[T]$, and $k = F(T)$. We will assume that the characteristic of F is not 2. Let $m \in A$ and define $|m| = q^{\deg(m)}$ if $m \neq 0$ and $|0| = 0$. If m is square-free, define $K_m = k(\sqrt{m})$ and $\mathcal{O}_m$ to be the integral closure of A in

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K_m. It is not hard to see that $\mathcal{O}_m = A[\sqrt{m}]$. Let ∞ denote the prime at infinity of k, and $S_m(\infty)$ the primes of K_m which lie above ∞. Then $\mathcal{O}_m$ can also be described as the elements of K_m whose only poles lie in $S_m(\infty)$.

The zeta function of A is defined by

$$\zeta_A(s) = \sum_a |a|^{-s} = \prod_p (1 - |p|^{-s})^{-1},$$

where the sum is over all monic elements $a \in A$ and the product is over all monic irreducibles $p \in A$. A straightforward calculation from the definition shows that $\zeta_A(s) = (1 - q^{(1-s)})^{-1}$. If m is a square free element of A, we set $\chi_m(a) = (m/a)$ where (m/a) is the Kronecker symbol, and define $L(s, \chi_m) = \sum_a \chi_m(a)|a|^{-s}$. Finally, we define the zeta function of the ring $\mathcal{O}_m$ so $\zeta_{\mathcal{O}_m}(s) = \sum_{\mathcal{A}} N\mathcal{A}^{-s}$ where $\mathcal{A}$ runs through the nonzero ideals of $\mathcal{O}_m$ and, as usual, $N\mathcal{A}$ is the number of elements in $\mathcal{O}_m/\mathcal{A}$. Just as in number fields, one has the relation

$$\zeta_{\mathcal{O}_m}(s) = \zeta_A(s)L(s, \chi_m).$$

We will use this fact together with results of Quillen and Tate to relate the number $L(2, \chi_m)$ to the size of the group $K_2(\mathcal{O}_m)$. First, we wish to recall one of the main results of [1] (see Theorem 0.8, as well as its generalization Theorem 5.2).

DEFINITION. For $s \in C$, $\mathrm{Re}(s) \geq 1/2$, define

$$c(s) = \prod_p (1 - |p|^{-2} - |p|^{-(2s+1)} + |p|^{-(2s+2)}).$$

It is easy to see that this product converges absolutely and uniformly. It is very close to $\zeta_A(2)^{-1}\zeta_A(2s + 1)^{-1}$.

THEOREM 1. *Let $\varepsilon > 0$ be given and assume $s \in C$ with* $\mathrm{Re}(s) \geq 1$.

(a) *If M is odd, $M = 2n + 1$, then*

$$(q - 1)^{-1}(q^M - q^{M-1})^{-1} \sum L(s, \chi_m) = \zeta_A(2)\zeta_A(2s)c(s) + O(q^{-n(1-\varepsilon)}),$$

where the sums is over all square-free m such that $\deg(m) = M$.

(b) *If M is even, $M = 2n$, then*

$$2(q - 1)^{-1}(q^M - q^{M-1})^{-1} \sum L(s, \chi_m) = \zeta_A(2)\zeta_A(2s)c(s) + O(q^{-n(1-\varepsilon)}),$$

where the sum is over either all square-free m of degree M with leading coefficient a square, or over all square-free m of degree M with leading coefficient a nonsquare.

Part (a) of the theorem corresponds to the polynomials m such that ∞ is ramified in K_m. The first subcase of part (b) corresponds to those m for which ∞ splits in K_m, and the second subcase corresponds to those m for which ∞ is inert in K_m.

It is an exercise to prove that the number of square-free polynomial of degree M is $(q - 1)(q^M - q^{M-1})$ so that the left-hand sides of the above equations are true average values of the L-functions under consideration. We remark that the factor of 2 on the left-hand side of the equation in part (b) is correct. In Theorem 0.8 of [1], it is mistakenly written as 1/2.

2

In this section we will discuss in more detail the groups $K_2(\mathcal{O}_m)$ and relate their size to the values $L(2, \chi_m)$.

Let K/F be a function field in one variable with a finite constant field F having q elements. We denote the primes in K by v and by $\mathcal{O}_v$ the valuation ring at v, by $\mathcal{P}_v$ its maximal ideal, and by $\bar{F}_v$ the residue class field at v. The tame symbol $(*, *)_v$ is a mapping from $K^* \times K^*$ to $\bar{F}_v^*$ defined by

$$(a, b)_v = (-1)^{v(a)v(b)} a^{v(b)}/b^{v(a)} \text{ modulo } \mathcal{P}_v.$$

This symbol is bimultiplicative and has the property that $(a, 1 - a)_v = 1$ for all $a \in K^*$ with $a \neq 0, 1$.

The group $K_2(K)$ can be defined as $K^* \otimes K^*$ modulo the subgroup generated by the elements $a \otimes (1 - a)$ for $a \in K^*$ with $a \neq 0, 1$. Clearly there is a natural map $\lambda_v : K_2(K) \to \bar{F}_v^*$ induced by $\lambda_v(a \otimes b) = (a, b)_v$. Let $\lambda : K_2(K) \to \oplus_v \bar{F}_v^*$ be the sum of the tame symbol maps, and $\mu : \oplus_v \bar{F}_v^* \to F^*$ be the map given by $\mu(\ldots, a_v, \ldots) = \prod_v a_v^{m_v/m}$ where $m_v = N\mathcal{P}_v - 1 = |\bar{F}_v^*|$ and $m = q - 1 = |F^*|$. Moore (see [5]) proved that the following sequence is exact.

$$(0) \to \ker(\lambda) \to K_2(K) \xrightarrow{\lambda} \bigoplus_v \bar{F}_v^* \xrightarrow{\mu} F^* \to (0).$$

In [5], Tate gives a proof of the Birch–Tate conjecture concerning the size of $\ker(\lambda)$. He proved that

$$|\ker(\lambda)| = (q - 1)(q^2 - 1)\zeta_K(-1),$$

where $\zeta_K(s) = \prod_v (1 - N\mathcal{P}_v^{-s})^{-1}$, the product being over all primes v of the function field K.

Now, let $S = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_t)$ be a finite set of primes of K, and denote by $\mathcal{O}_S$ the ring of S-integers of K, i.e., the elements of K whose poles lie in S. Using a theorem of Quillen [3], we have an exact sequence

$$(0) \to K_2(\mathcal{O}_S) \to K_2(K) \xrightarrow{\lambda'} \bigoplus_{v \notin S} \bar{F}_v^* \to (0).$$

The map λ' is the truncation of the map λ. For those readers not familiar with the definition of K_2 of a ring, one does not lose much in taking $\ker(\lambda')$ as the definition of $K_2(\mathcal{O}_S)$. For those who would like to see this material in a fuller context, the survey article [4] by Ramakrishnan is recommended. Our next task is to compute the order of $K_2(\mathcal{O}_S)$. We need a definition.

DEFINITION. With S defined as above we define the S-zeta function of K to be

$$\zeta_S(s) = \prod_{v \notin S} (1 - N\mathcal{P}_v^{-s})^{-1}.$$

PROPOSITION 1. $|K_2(\mathcal{O}_S)| = (-1)^t(q^2 - 1)\zeta_S(-1)$.

Before giving the proof of this Proposition we state and prove an immediate corollary.

COROLLARY. *Let $A = F[T]$, $k = F(T)$, and $K_m = k(\sqrt{m})$ where $m \in A$ is a square-free polynomial of positive degree. Let $\mathcal{O}_m = A[\sqrt{m}]$. Then, $|K_2(\mathcal{O}_m)| = (-1)^{t-1}L(-1, \chi_m)$ where t is the number of primes in K_m above ∞.*

Proof. Let $S = S_m(\infty)$ be the set of primes in K_m above ∞. Then, $\mathcal{O}_S$ is just $\mathcal{O}_m$, so by the Proposition, $|K_2(\mathcal{O}_m)| = (-1)^t(q^2 - 1)\zeta_S(-1)$. Since, $\zeta_S(s) = \zeta_A(s)L(s, \chi_m))$ and $\zeta_A(s) = (1 - q^{(1-s)})^{-1}$, the corollary follows.

Proof of the Proposition 1. One begins by forming the following commutative diagram:

$$\begin{array}{ccccccccc}
(0) & \to & K_2(\mathcal{O}_S) & \to & K_2(K) & \xrightarrow{\lambda'} & \bigoplus_{v \notin S} \bar{F}_v^* & \to & (0) \\
 & & {\scriptstyle i}\uparrow & & {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle \alpha}\uparrow & & \\
(0) & \to & \ker(\lambda) & \to & K_2(K) & \to & \mathrm{im}(\lambda) & \to & (0).
\end{array}$$

Here, i is inclusion, and α is induced by truncation. By the 5-lemma, $|\ker(\alpha)| = |K_2(\mathcal{O}_S)/i(\ker(\lambda)|$. We have to compute $|\ker(\alpha)|$. Now, $(a_v) \in \ker(\alpha)$ if and only if $a_v = 1$ for $v \notin S$ and $\prod_{v \in S} a_v^{m_v/m} = 1$. Thus it is seen that $\ker(\alpha)$ is the kernel of an onto map from $\prod_{v \in S} \bar{F}_v^*$ to F^*. It follows that

$$|\ker(\alpha)| = (q - 1)^{-1} \prod_{i=1}^{t} (N\mathcal{P}_i - 1).$$

To conclude, one calculates

$$\begin{aligned}|K_2(\mathcal{O}_S)| &= |\ker(\lambda)|\,|\ker(\alpha)| \\ &= (q-1)(q^2-1)\zeta_K(-1)(q-1)^{-1}\prod_{i=1}^{t}(N\mathcal{P}_i - 1) \qquad (1)\\ &= (-1)^t(q^2-1)\zeta_S(-1).\end{aligned}$$

The last thing we have to do in this section is to find the exact relationship between the numbers $L(-1, \chi_m)$ and $L(2, \chi_m)$. We have two equations

$$\zeta_{\mathcal{O}_m}(s) = L(s, \chi_m)\zeta_A(s) \qquad \text{and} \qquad \zeta_{K_m}(s) = \tilde{L}_m(s)\zeta_k(s). \qquad (2)$$

From these two equations and the fact that $\zeta_k(s) = (1 - q^{-s})^{-1}(1 - q^{(1-s)})^{-1}$, one deduces the following relationship between $L(s, \chi_m)$ and $\tilde{L}_m(s)$:

$$\tilde{L}_m(s) = L(s, \chi_m)(1 - q^{-s}) \prod_{v \in S_m(\infty)} (1 - N\mathcal{P}_v^{-s})^{-1}. \qquad (3)$$

The function $\tilde{L}_m(s)$ satisfies the well-known functional equation $\tilde{L}_m(1 - s) = q^{-g}q^{2gs}\tilde{L}_m(s)$ (see Thm. 4 in Chap. VII of [6]). The number g is the genus of the function field K_m. Let M be the degree of m. Since we are assuming that m is square-free it is standard that the genus of the function field defined by $y^2 = m$ is given by $g = (M/2) - 1$ if M is even, and by $g = (M - 1)/2$ if M is odd.

We have now assembled everything we need to prove the following proposition.

PROPOSITION 2. *Let $K_m = k(\sqrt{m})$ where m is a square-free polynomial of degree M. Then,*

(a) *If M is odd,*

$$|K_2(\mathcal{O}_m)| = q^{(3/2)M}q^{-3/2}L(2, \chi_m).$$

(b) *If M is even and the leading coefficient of m is a square, then*

$$|K_2(\mathcal{O}_m)| = q^{(3/2)M}(q^2 + q)^{-1}L(2, \chi_m).$$

(c) *If M is even and the leading coefficient of m is not a square, then*

$$|K_2(\mathcal{O}_m)| = q^{(3/2)M}(q + 1)q^{-1}(q^2 + 1)^{-1}L(2, \chi_m).$$

Proof. As mentioned several times before, cases (a), (b), and (c), of the theorem corresponds to ∞ ramifying, splitting, or being inert in K_m. Thus,

$$\prod_{v \in S_m(\infty)} (1 - N\mathcal{P}_v^{-s})^{-1} = (1 - q^{-s})^{-1}, (1 - q^{-s})^{-2}, (1 - q^{-2s})^{-1}$$

respectively in the three cases. This allows us to give the exact relation between $L(-1, \chi_m)$ and $\tilde{L}_m(-1)$ by using Eq. (3), and via the functional equation satisfied by $\tilde{L}_m(s)$ to relate these values to those at $s = 2$.

In case (a), we have $L(s, \chi_m) = \tilde{L}_m(s)$. The genus in this case is $(M - 1)/2$ so by the functional equation $\tilde{L}_m(-1) = q^{3(M-1/2)}\tilde{L}_m(2)$. It follows that $L(-1, \chi_m) = q^{M/2}q^{-3/2}L(2, \chi_m)$ and part (a) is now immediate from the corollary to Proposition 1.

In case (b), we have by Eq. (3) that $L(s, \chi_m) = (1 - q^{-s})\tilde{L}_m(s)$. In this case the genus is $(M/2) - 1$, so by the functional equation we have $L(-1, \chi_m) = (1 - q)\tilde{L}_m(-1) = (1 - q)q^{3((M/2)-1)}\tilde{L}_m(2) = (1 - q)q^{3(M/2)}q^{-3}(1 - q^{-2})L(2, \chi_m)$. Simplifying and using the corollary to Proposition 1 once again, part (b) follows.

In case (c), we have by Eq. (3) that $L(s, \chi_m) = (1 + q^{-s})\tilde{L}_m(s)$. The genus is again $M/2 - 1$ so by the functional equation $L(-1, \chi_m) = (1 + q)\tilde{L}_m(-1) = (1 + q)q^{3((M/2)-1)}\tilde{L}_m(2) = (1 + q)q^{3(M/2)}q^{-3}(1 + q^{-2})^{-1}L(2, \chi_m)$. Simplifying, and using the corollary to Proposition 1 one last time yields part (c).

3

By combining Theorem 1 with Proposition 2 we can now state our main result.

MAIN THEOREM. *Let $\varepsilon > 0$ be given.*

(a) *Suppose M is odd. Then the average value of $|K_2(\mathbb{O}_m)|$ as m varies over all square-free polynomials of degree M is given by*

$$\zeta_A(2)\zeta_A(4)c(2)q^{-3/2}q^{3(M/2)} + O(q^{M(1+\varepsilon)}).$$

(b) *Suppose M is even. Then the average value of $|K_2(\mathbb{O}_m)|$ as m varies over all square-free polynomials of degree M with leading coefficient a square is given by*

$$\zeta_A(2)\zeta_A(4)(q^2 + q)^{-1}c(2)q^{3(M/2)} + O(q^{M(1+\varepsilon)}).$$

(c) *Suppose M is even. Then the average value of* $|K_2(\mathbb{O}_m)|$ *as m varies over all square-free polynomials of degree M whose leading coefficient is not a square is given by*

$$\zeta_A(2)\zeta_A(4)(q + 1)q^{-1}(q^2 + 1)^{-1}c(2)q^{3(M/2)} + O(q^{M(1+\varepsilon)}).$$

The constant $c(2)$ which occurs in all three cases is given by

$$c(2) = \prod_p (1 - |p|^{-2} - |p|^{-5} + |p|^{-6}).$$

If we replace the number 6 by 5, we see that $c(2)$ is very close to $\zeta_A(2)^{-1}\zeta_A(5)^{-1}$. Thus the expression $\zeta_A(2)\zeta_A(4)c(2)$ which occurs in all three cases is close to $\zeta_A(4)/\zeta_A(5)$, which is reminiscent of the expression $\zeta_A(2)/\zeta_A(3)$ that occurs in connection with average values of class numbers (see [1]).

Finally, we note that the order of magnitude of the average values occuring in the theorem, namely $q^{3(M/2)}$, is predictable from Tate's theorem and the form of the functional equation. The new element here is the determination of the constant that goes in front of $q^{3(M/2)}$ and the size of the error term.

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